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# Identical particle states in the probability representation of quantum mechanics

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## Abstract

We study the symmetry properties under permutation of tomograms describing the states of a system of identical particles in the probability representation. Starting from the action of the permutation group on the density matrix, we define its action on the tomographic probability distribution. Explicit calculations are performed for the two-dimensional harmonic oscillator.

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## 1. Introduction

In the Schrödinger approach to quantum mechanics [1], the peculiarities of the quantum behaviour of a system of identical particles are encoded in the symmetry properties of the wavefunctions. Analogously, the description in terms of density operators [2] implies, for the density matrices associated with the states of identical particles, definite symmetry properties. These two approaches, which are probably those most commonly used among physicists, are very different from the approach of classical statistical mechanics in terms of probability distributions defined on the phase space of the system. There is another point of view, which has developed almost simultaneously with those mentioned, that responds to the wish to render the quantum picture closer to the classical statistics approach. This has given rise to the introduction of the Wigner quasi-distribution functions on the phase space [3] obeying the Moyal equation [4] and to hidden variable theories [5]. The aim of describing quantum states in terms of a classical distribution of probability then determined the introduction of another class of quasi-distributions for the quantum states [6–8]. In 1969, Cahill and Glauber [9] considered a set of s-ordered quasi-distribution functions that further generalized the quasi-probability distributions previously introduced by Wigner, Husimi, Glauber and Sudarshan.

They showed that for each quantum state it is possible to find some probability distribution which is obtained in terms of the density operator of the given state. But it was not clear whether or not such a probability distribution determined the density operator uniquely, that is, whether or not the map was invertible.

Only recently was it realized that quantum mechanics could be described completely in terms of standard probability distributions suitably defined for a random variable [10]. In [10], a consistent scheme has been proposed, the so-called probability representation, whose results are completely equivalent, in the sense of invertibility, to the ordinary formulation of quantum mechanics. The quantum states are described by a tomographic distribution of probability, or tomogram, also known as a marginal distribution function (MDF), and the evolution of the system is described by an integro-differential equation of generalized Fokker–Planck-type for the MDF. The problem of diffraction in time posed and studied in [11] was considered both in the Wigner representation and in the probability representation of quantum states in [12]. The probability representation of quantum mechanics uses as a mathematical tool the symplectic tomography map [13, 14] of density operators on to quadrature probability distributions. For a general approach to tomograms and quasi-distributions of quantum states, see also [15, 16] and, with the inclusion of spin, [17–20].

Recently, in [21] nine existing approaches to quantum mechanics have been reviewed: the wavefunction, matrix, path integral [22], phase space, density matrix, second quantization, variational, pilot wave and Hamilton–Jacobi formulation. All these approaches are equivalent, and each helps to better understand different aspects of quantum mechanics. For example, the Feynman path integral framework is the one that better illustrates the connection between the Green function of the Schrödinger equation and the classical action. The phase space formulation provides an understanding of the relation with classical statistics, whereas the density matrix is the most suitable to describe entanglement phenomena, and so on. The various formulations may look very different at a first sight, so that efforts have been necessary (and in some cases they are still continuing) to show that all the physical aspects of quantum phenomena, including identical particle effects, can be elaborated in all the existing formulations.

The new formulation of quantum mechanics is complementary to those already available, since it allows us to understand quantum phenomena from yet another point of view. Albeit the equivalence is proven through the one-to-one map with the density matrix, it is necessary to rediscover all the results available in the other formulations to achieve a satisfactory status for the new theory. Of course, it may well happen that this will not be a better point of view to describe one phenomenon or another, exactly as happens in the other frameworks.

Among other things, we need to demonstrate that the effects of identical particle statistics can also be explained in the framework of the probability representation, as shown for the existing nine formulations. Identical particles obey either Bose or Fermi statistics. As a result, the wavefunction describing the system must be symmetrized or antisymmetrized, respectively. Because the states of a physical system belong to a vector space and are, consequently, linearly superposable, it becomes quite easy to implement the two types of statistics in the ordinary quantum mechanics considering, for example, the wavefunctions of the system as a basis for a representation of the permutation group. On the other hand, in the probability representation of quantum states we need to know the behaviour of the tomographic probability distributions for systems of identical particles. The aim of our paper is the formulation of the symmetry properties of tomograms for systems of identical particles.

In the following, we use the relation between the MDF and the density matrix to construct the action of the permutation group on the MDFs. Because the set of MDFs is not a vector space (see, for example, [23] where a superposition principle was formulated for both sets

of density matrices and tomograms) we obtain a realization, and not a representation, of the permutation group on the MDFs. In this way we can introduce the completely symmetrized and antisymmetrized MDFs.

The paper is organized as follows. In section 2 we review the realization of the permutation group on the set of density matrices, and in section 3 we derive the corresponding realization on the set of MDFs. For the sake of clarity, we work explicitly on the case of two identical particles and then we extend the results to  $n$  particles in section 4. In section 5 we describe a simple application and finally we conclude with some brief remarks.

## 2. The permutation group of the density matrix

Identical particles obey either Bose or Fermi statistics. This property implies two specific kinds of behaviour of the wavefunction of two identical particles: it must be symmetric in the case of Bose particles, antisymmetric for Fermi particles. These properties are more precisely described by means of the permutation group representation theory. To obtain a model, let us concentrate on the system of two one-dimensional particles whose positions are  $x_1$  and  $x_2$ , respectively. The corresponding wavefunction,  $\psi(x_1, x_2)$ , can be decomposed into the sum of symmetric and antisymmetric functions:

$$\begin{aligned}\psi(x_1, x_2) &= \frac{1}{2}(\psi(x_1, x_2) + \psi(x_2, x_1)) + \frac{1}{2}(\psi(x_1, x_2) - \psi(x_2, x_1)) \\ &= \psi_+(x_1, x_2) + \psi_-(x_1, x_2).\end{aligned}\quad (2.1)$$

This decomposition may be related to the irreducible representation of the permutation group  $G = (\hat{E}, \hat{p}_{21})$ , where  $\hat{E}$  is the identity of the group and  $\hat{p}_{21}$  represents the operation of permutation of the coordinates  $x_1$  and  $x_2$ , i.e.

$$\hat{E}\psi(x_1, x_2) = \psi(x_1, x_2) \quad \hat{p}_{21}\psi(x_1, x_2) = \psi(x_2, x_1).\quad (2.2)$$

The table of characters of the representations of the permutation group of two elements has the form

$$\begin{array}{cc} \hat{E} & \hat{p}_{21} \\ 1 & 1 \\ 1 & -1. \end{array}\quad (2.3)$$

This means that the decomposition (2.1) is connected with the irreducible representations (2.2) through the following formulae:

$$\hat{E}\psi_{\pm}(x_1, x_2) = +1\psi_{\pm}(x_1, x_2) \quad \hat{p}_{21}\psi_{\pm}(x_1, x_2) = \pm 1\psi_{\pm}(x_1, x_2).\quad (2.4)$$

Thus, the two functions  $\psi_+$  and  $\psi_-$  realize a basis of the one-dimensional representation of the permutation group. We say that identical particles with symmetric and antisymmetric wavefunctions are described by means of the symmetric and antisymmetric representations of  $G$ , respectively.

The density matrix  $\rho(x_1, x_2, x'_1, x'_2)$  of a pure state of two particles with wavefunction  $\psi(x_1, x_2)$  has the form

$$\rho(x_1, x_2, x'_1, x'_2) = \psi(x_1, x_2)\psi^*(x'_1, x'_2).\quad (2.5)$$

Thus, we can extend the action of the permutation group to the set of density matrices because the coordinates  $x_1, x_2$  and  $x'_1, x'_2$  can be permuted independently. This means that the group  $\tilde{G} = G \otimes G$ , which is the direct product of the permutation groups of two elements, is that related to the symmetry properties of the density matrix.  $\tilde{G}$  contains four elements:

$$\hat{I} = \hat{E} \otimes \hat{E} \quad \hat{P}_{21} = \hat{p}_{21} \otimes \hat{E} \quad \hat{P}'_{21} = \hat{E} \otimes \hat{p}_{21} \quad \hat{\Pi}_{21} = \hat{p}_{21} \otimes \hat{p}_{21}.\quad (2.6)$$

The action of the four elements on the density matrix (2.5) is defined as follows:

$$\hat{I}\rho(x_1, x_2, x'_1, x'_2) = \rho(x_1, x_2, x'_1, x'_2) \quad (2.7)$$

$$\hat{P}_{21}\rho(x_1, x_2, x'_1, x'_2) = \rho(x_2, x_1, x'_1, x'_2) \quad (2.8)$$

$$\hat{P}'_{21}\rho(x_1, x_2, x'_1, x'_2) = \rho(x_1, x_2, x'_2, x'_1) \quad (2.9)$$

$$\hat{\Pi}_{21}\rho(x_1, x_2, x'_1, x'_2) = \rho(x_2, x_1, x'_2, x'_1). \quad (2.10)$$

Using the multiplication table of group  $\tilde{G}$

	$\hat{I}$	$\hat{P}_{21}$	$\hat{P}'_{21}$	$\hat{\Pi}_{21}$
$\hat{I}$	$\hat{I}$	$\hat{P}_{21}$	$\hat{P}'_{21}$	$\hat{\Pi}_{21}$
$\hat{P}_{21}$	$\hat{P}_{21}$	$\hat{I}$	$\hat{\Pi}_{21}$	$\hat{P}'_{21}$
$\hat{P}'_{21}$	$\hat{P}'_{21}$	$\hat{\Pi}_{21}$	$\hat{I}$	$\hat{P}_{21}$
$\hat{\Pi}_{21}$	$\hat{\Pi}_{21}$	$\hat{P}'_{21}$	$\hat{P}_{21}$	$\hat{I}$

we may construct a realization through the action of  $\tilde{G}$  on the following distributions:

$$\rho_+ = \frac{1}{4}(\hat{I}\rho + \hat{\Pi}\rho + \hat{P}_{21}\rho + \hat{P}'_{21}\rho) \quad (2.11)$$

$$\rho_- = \frac{1}{4}(\hat{I}\rho + \hat{\Pi}\rho - \hat{P}_{21}\rho - \hat{P}'_{21}\rho) \quad (2.12)$$

$$\rho_1 = \frac{1}{4}(\hat{I}\rho - \hat{\Pi}\rho + \hat{P}_{21}\rho - \hat{P}'_{21}\rho) \quad (2.13)$$

$$\rho_2 = \frac{1}{4}(\hat{I}\rho - \hat{\Pi}\rho - \hat{P}_{21}\rho + \hat{P}'_{21}\rho). \quad (2.14)$$

It is easy to see that such an action is given by

	$\hat{I}$	$\hat{P}_{21}$	$\hat{P}'_{21}$	$\hat{\Pi}_{21}$
$\rho_+$	1	1	1	1
$\rho_-$	1	-1	-1	1
$\rho_1$	1	1	-1	-1
$\rho_2$	1	-1	1	-1

Incidentally we note that  $\rho_+$  and  $\rho_-$  are also obtained as

$$\begin{aligned} \rho_+(x_1, x_2, x'_1, x'_2) &= \psi_+(x_1, x_2)\psi_+^*(x'_1, x'_2) \\ &= \frac{1}{4}[(\psi(x_1, x_2)\psi^*(x'_1, x'_2) + \psi(x_2, x_1)\psi^*(x'_1, x'_2) \\ &\quad + \psi(x_1, x_2)\psi^*(x'_2, x'_1) + \psi(x_2, x_1)\psi^*(x'_2, x'_1)] \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} \rho_-(x_1, x_2, x'_1, x'_2) &= \psi_-(x_1, x_2)\psi_-^*(x'_1, x'_2) \\ &= \frac{1}{4}[\psi(x_1, x_2)\psi^*(x'_1, x'_2) - \psi(x_2, x_1)\psi^*(x'_1, x'_2) \\ &\quad - \psi(x_1, x_2)\psi^*(x'_2, x'_1) + \psi(x_2, x_1)\psi^*(x'_2, x'_1)]. \end{aligned} \quad (2.16)$$

That is,  $\rho_+$  and  $\rho_-$  are the usual density matrices respectively associated with the usual symmetric and antisymmetric wavefunctions. (Here, and throughout the paper, everything is obtained for pure states but because of the linearity of the mixtures of pure states we can easily generalize to the density matrices of the arbitrary mixed states of two particles.)

Thus, what we have done so far is to extend the action of the permutation group (which we know for the space of the wavefunctions) to the space of density matrices (equations (2.7)–(2.10)) by means of equation (2.5) which links in a simple way the density matrices to the

wavefunctions. The natural generalization of this procedure, i.e. the definition of the action of the permutation group on the space of tomograms through the expression of tomograms in terms of density matrices, is not so simple since such a relation is an integral transform. This implies that the density matrices and the MDFs depend on different variables and, as we see in the next section, it is not clear what the action is of the permutation group on the new variables. We then follow a different approach which we illustrate first for the space of the density matrices. What we are looking for is an expression of the symmetrized and antisymmetrized density matrices in integral form and, more precisely, we seek a realization of the permutation group in terms of integral kernels. Starting from equations (2.7) and (2.10) it is easy to realize that

$$\begin{aligned} \rho_{\pm} &= \frac{1}{4} \{ \rho(x_1, x_2, x'_1, x'_2) + \rho(x_1, x_2, x'_2, x'_1) \pm [ \rho(x_2, x_1, x'_1, x'_2) + \rho(x_1, x_2, x'_2, x'_1) ] \} \\ &= \frac{1}{4} \int dy_1 dy_2 dy'_1 dy'_2 \{ \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y'_1, x'_1} \delta_{y'_2, x'_2} + \delta_{y_1, x_2} \delta_{y_2, x_1} \delta_{y'_1, x'_2} \delta_{y'_2, x'_1} \\ &\quad \pm [ \delta_{y_1, x_2} \delta_{y_2, x_1} \delta_{y'_1, x'_1} \delta_{y'_2, x'_2} + \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y'_1, x'_2} \delta_{y'_2, x'_1} ] \} \rho(y_1, y_2, y'_1, y'_2) \end{aligned} \quad (2.17)$$

with  $\delta_{y_i, x_j} = \delta(y_i - x_j)$ , that is

$$\rho_{\pm}(x_1, x_2; x'_1, x'_2) = \int K_{\pm}(x_1, x_2, x'_1, x'_2; y_1, y_2, y'_1, y'_2) \rho(y_1, y_2; y'_1, y'_2) dy_1 dy_2 dy'_1 dy'_2 \quad (2.18)$$

with the kernel

$$\begin{aligned} K_{\pm}(x_1, x_2, x'_1, x'_2; y_1, y_2, y'_1, y'_2) &= \frac{1}{4} \{ \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y'_1, x'_1} \delta_{y'_2, x'_2} + \delta_{y_1, x_2} \delta_{y_2, x_1} \delta_{y'_1, x'_2} \delta_{y'_2, x'_1} \\ &\quad \pm [ \delta_{y_1, x_2} \delta_{y_2, x_1} \delta_{y'_1, x'_1} \delta_{y'_2, x'_2} + \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y'_1, x'_2} \delta_{y'_2, x'_1} ] \}. \end{aligned} \quad (2.19)$$

Defining the permutation operator as

$$P_{j_1, j_2, \dots, j_n} f(x_1, x_2, \dots, x_n) = f(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \quad (2.20)$$

we can write the kernel as

$$K_{\pm}(x, x' | y, y')_2 = \frac{1}{(2!)^2} (I \pm P_{21}) \otimes (I \pm P'_{21}) \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y'_1, x'_1} \delta_{y'_2, x'_2} \quad (2.21)$$

where  $(x, x' | y, y')_2 \equiv (x_1, x_2, x'_1, x'_2; y_1, y_2, y'_1, y'_2)$  and we assume that  $P$  and  $P'$  act on  $x$  and  $x'$ , respectively. Analogously we may define the kernels  $K_1$  and  $K_2$  associated with the density matrices  $\rho_1$  and  $\rho_2$ . In this way we obtain a realization of the permutation group in terms of integral kernels. This approach is absolutely equivalent to that previously described but it is the one we need in order to implement the symmetry properties in the MDF framework.

### 3. The permutation group of the MDF

We are now in a position to extend the previous procedure to the tomographic probability distribution, but we first need a brief review of the theory.

The MDF of a random variable  $X$  is defined in [9] as the Fourier transform of the quantum characteristic function  $\chi(k) = \langle e^{ik\hat{X}} \rangle$

$$w(X, t) = \frac{1}{2\pi} \int dk e^{-ikX} \langle e^{ik\hat{X}} \rangle \quad (3.1)$$

where  $\hat{X}$  is the operator associated with  $X$  and, for each observable  $\hat{O}$ ,  $\langle \hat{O} \rangle = \text{Tr}(\hat{\rho} \hat{O})$  with  $\hat{\rho}$  being the time-dependent density operator. The MDF so defined is positive and normalized to unity, provided  $\hat{X}$  is an observable [9].

In [14] it is shown that by taking

$$X = \mu q + \nu p \quad (3.2)$$

where  $q$  and  $p$  are two conjugate variables and  $\mu$  and  $\nu$  are real parameters labelling different reference frames in phase space,  $w(X, \mu, \nu, t)$  is normalized with respect to the  $X$  variable and there exists an invertible relation between the MDF and the density matrix. The variable  $X$  represents the position coordinate taking values in an ensemble of reference frames. Equation (3.1) may be rewritten in the more convenient form [24]

$$w(X, \mu, \nu, t) = \frac{1}{2\pi|\nu|\hbar} \int \rho(Z', Z'', t) \exp \left[ -i \frac{Z' - Z''}{\nu\hbar} \left( X - \mu \frac{Z' + Z''}{2} \right) \right] dZ' dZ'' \quad (3.3)$$

and its inverse is represented by

$$\rho(X, X', t) = \frac{1}{2\pi} \int w(Y, \mu, X - X', t) \exp \left[ \frac{i}{\hbar} \left( Y - \mu \frac{X + X'}{2} \right) \right] d\mu dY. \quad (3.4)$$

It is important to note that, for equation (3.3) to be invertible, it is necessary that  $X$  be a coordinate variable taking values in an ensemble of phase spaces. In other words, the specific choice  $\mu = 1, \nu = 0$  or any other fixing of the parameters  $\mu$  and  $\nu$  would not allow us to reconstruct the density matrix. Hence, the MDF contains the same amount of information on a quantum state as the density matrix, only if equation (3.2) is assumed.

For Hamiltonians of the form

$$H = \frac{p^2}{2m} + V(q) \quad (3.5)$$

an evolution equation governing the time dependence of the MDF is available (see [24] for a simple derivation)

$$\begin{aligned} \partial_t w(X, \mu, \nu, t) = & \left\{ \frac{\mu}{m} \partial_\nu + \frac{i}{\hbar} \left[ V \left( -(\partial_X)^{-1} \partial_\mu - \frac{i\nu\hbar}{2} \partial_X \right) \right. \right. \\ & \left. \left. - V \left( -(\partial_X)^{-1} \partial_\mu + \frac{i\nu\hbar}{2} \partial_X \right) \right] \right\} w(X, \mu, \nu, t) \end{aligned} \quad (3.6)$$

where the inverse derivative is defined as

$$(\partial_X)^{-1} \int f(Z) e^{g(Z)X} dZ = \int \frac{f(Z)}{g(Z)} e^{g(Z)X} dZ. \quad (3.7)$$

The evolution equation, of the generalized Fokker–Planck type, plays the role of the Schrödinger equation in the alternative scheme we are outlining. Its classical limit is easily seen to be

$$\dot{w}(X, \mu, \nu, t) = \left\{ \frac{\mu}{m} \frac{\partial}{\partial \nu} + \nu V' \left( - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} \right) \frac{\partial}{\partial X} \right\} w(X, \mu, \nu, t) \quad (3.8)$$

where  $V'$  is the derivative of the potential with respect to the argument. Equation (3.8) may be checked to be equivalent to the Boltzmann equation for a classical distribution of probability  $f(q, p, t)$ ,

$$\frac{\partial f}{\partial t} + \frac{p}{m} \frac{\partial f}{\partial q} - \frac{\partial V}{\partial q} \frac{\partial f}{\partial p} = 0 \quad (3.9)$$

after performing the change of variables

$$w(X, \mu, \nu, t) = \frac{1}{2\pi} \int f(q, p, t) e^{ik(X - \mu q - \nu p)} dk dq dp. \quad (3.10)$$

Hence, the classical and quantum evolution equations only differ by terms of higher order in  $\hbar$ . Moreover, for potentials quadratic in  $\hat{q}$ , higher-order terms cancel out and the quantum evolution equation coincides with the classical one. This leads to the remarkable result that there is no difference between the evolution of the distributions of probability for quantum and classical observables, when the system is described by a Hamiltonian quadratic in positions and momenta. The generalization to  $N$  particles and eventually to field theory is straightforward and may be found in [24].

The scheme just outlined is self-consistent and does not require at any step external structures such as the wavefunction or the density matrix. In this sense, the probability description of quantum mechanics in terms of tomograms, once completely formulated, has to furnish a perfectly equivalent scheme to conventional ones [10]. Thus, it appears quite natural to investigate the symmetry properties of the MDF of identical particles with respect to permutations. In this regard, we discuss for simplicity the case of two particles, while a generalization is exhibited in the next section.

From equation (3.3) the MDF of two identical particles is represented by

$$w(\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2) = k_2 \int D_2(z; z') e^{-i[\alpha_1 + \alpha_2]} \rho(z_1, z_2; z'_1, z'_2) \quad (3.11)$$

where we have defined

- (i)  $\alpha_i = \frac{z_i - z'_i}{|v_i| \hbar} \left[ \xi_i - \mu_i \frac{z_i + z'_i}{2} \right]$ ,
- (ii)  $D_n(z; z'; x; \dots; y) = \left( \prod_{i=1}^n dz_i \right) \left( \prod_{i=1}^n dz'_i \right) \left( \prod_{i=1}^n dx_i \right) \cdots \left( \prod_{i=1}^n dy_i \right)$ ,
- (iii)  $k_n \equiv \prod_{i=1}^n \frac{1}{2\pi \hbar |v_i|}$ ;

and the time dependence has been omitted. We note that if

$$\rho(z_1, z_2; z'_1, z'_2) = \rho_1(z_1, z'_1) \rho_2(z_2, z'_2) \quad (3.12)$$

the tomogram of a system of two particles factorizes as well

$$w(\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2) = w_1(\xi_1, \mu_1, \nu_1) w_2(\xi_2, \mu_2, \nu_2) \quad (3.13)$$

with

$$w_i(\xi_i, \mu_i, \nu_i) = \frac{1}{2\pi \hbar |v_i|} \int dz_i dz'_i e^{-i\alpha_i} \rho_i(z_i, z'_i). \quad (3.14)$$

Therefore, it seems natural to identify the MDF relative to symmetrized and antisymmetrized states as

$$w_{\pm}(\xi, \mu, \nu)_2 := k_2 \int D_2(z; z') e^{-i[\alpha_1 + \alpha_2]} \rho_{\pm}(z|z')_2 \quad (3.15)$$

with  $(\xi, \mu, \nu)_n \equiv (\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2; \dots \xi_n, \mu_n, \nu_n)$ . In the same way we may define  $w_1$  and  $w_2$  as tomograms associated with  $\rho_1$  and  $\rho_2$ .

From equation (2.18), which represents  $\rho_{\pm}$  in terms of the integral kernel, we can write equation (3.15) in the form

$$w_{\pm}(\xi, \mu, \nu)_2 = k_2 \int D_2(z, y; z', y') K_{\pm}(y, y'|z, z')_2 e^{-i[\alpha_1 + \alpha_2]} \rho(y|y')_2. \quad (3.16)$$

Then, using the inverse formula (3.4) for the case of two particles

$$\rho(y|y')_2 = \frac{1}{(2\pi)^2} \int D_2(m, x) e^{i[\beta_1 + \beta_2]} w(x, m, y - y')_2 \quad (3.17)$$



with  $\beta_i = x_i - m_i \frac{y_i+y'_i}{2}$ , we obtain

$$\begin{aligned}
 w_{\pm}(\xi, \mu, \nu)_2 &= \frac{k_2}{(2\pi)^2} \int D_2(y, y', z, z', x, m, n) \delta_{n_1, y_1-y'_1} \delta_{n_2, y_2-y'_2} \\
 &\quad \times e^{-i[\alpha_1+\alpha_2]} e^{i[\beta_1+\beta_2]} K_{\pm}(y, y'; z, z')_2 w(x, m, n)_2 \\
 &= \int D_2(x, m, n) \left\{ \frac{k_2}{(2\pi)^2} \int D_2(y, y', z, z') \delta_{n_1, y_1-y'_1} \delta_{n_2, y_2-y'_2} \right. \\
 &\quad \left. \times e^{-i[\alpha_1+\alpha_2]} e^{i[\beta_1+\beta_2]} K_{\pm}(y, y'; z, z')_2 \right\} w(x, m, n)_2. \tag{3.18}
 \end{aligned}$$

Thus, the tomographic probability distribution corresponding to symmetric or antisymmetric states of two particles may be finally written as

$$w_{\pm}(\xi, \mu, \nu)_2 = \int D_2(x, m, n) \tilde{K}_{\pm}(\xi, \mu, \nu|x, m, n)_2 w(x, m, n)_2 \tag{3.19}$$

with the kernel  $\tilde{K}_{\pm}$  given by

$$\begin{aligned}
 \tilde{K}_{\pm}(\xi, \mu, \nu|x, m, n)_2 &= \frac{k_2}{(2\pi)^2} \int D_2(y, y'; z, z') \delta_{n_1, y_1-y'_1} \delta_{n_2, y_2-y'_2} \\
 &\quad \times \exp[-i(\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2)] K_{\pm}(y, y'|z, z')_2. \tag{3.20}
 \end{aligned}$$

Also in this case it is straightforward, but not particularly illuminating, to derive the integral kernels  $\tilde{K}_1$  and  $\tilde{K}_2$ , associated with  $w_1$  and  $w_2$ , which complete the realization of the permutation group on the space of tomograms.

#### 4. Generalization to $N$ particles

The generalization to  $N$  particles is now straightforward. We define the kernel  $K_{\pm}(y, y'; z, z')_n$  as follows

$$K_{\pm}(y, y'; z, z')_n := \frac{1}{(n!)^2} \sum_P \epsilon_P P_{j_1 \dots j_n} \otimes \sum_{P'} \epsilon_{P'} P'_{j_1 \dots j_n} \delta_{y_1, x_1} \dots \delta_{y_n, x_n} \delta_{y'_1, x'_1} \delta_{y'_n, x'_n} \tag{4.1}$$

where  $\epsilon_P$  and  $\epsilon_{P'}$  are equal to 1 in the symmetric case, while representing, in the antisymmetric case, the signs of the permutations  $P$  and  $P'$ , respectively. Therefore, we obtain

$$\tilde{K}_{\pm}(\xi, \mu, \nu)_n = \frac{k_n}{(2\pi)^n} \int D_n(y, y', z, z') \left[ \prod_{j=1}^n \delta_{n_j, y_j-y'_j} e^{-i\alpha_j} e^{i\beta_j} \right] K_{\pm}(y, y'; z, z')_n \tag{4.2}$$

so that we find

$$w_{\pm}(\xi, \mu, \nu)_n = \int D_n(t, m, n) \tilde{K}_{\pm}(\xi, \mu, \nu|t, m, n)_n w(x, m, n)_n. \tag{4.3}$$

We may introduce a sort of generalized Slater determinant [25] to obtain  $K_{\pm}$  and, by means of equation (4.2),  $\tilde{K}_{\pm}$ . We define

$$\Delta_{\pm}(y|x)_n = \frac{1}{n!} \left\| \begin{array}{cccc} \delta_{y_1, x_1} & \delta_{y_1, x_2} & \dots & \delta_{y_1, x_n} \\ \delta_{y_2, x_1} & \delta_{y_2, x_2} & \dots & \delta_{y_2, x_n} \\ \vdots & \vdots & & \vdots \\ \delta_{y_n, x_1} & \delta_{y_n, x_2} & \dots & \delta_{y_n, x_n} \end{array} \right\| \tag{4.4}$$

with the convention that for the symmetric case we always take the plus sign in computing the determinant. In this way we can write the kernel  $K_{\pm}(y, y'|z, z')$  in a more conventional manner as

$$K_{\pm}(y, y'; z, z')_n := \Delta_{\pm}(y|z)_n \Delta_{\pm}(y'|z')_n. \tag{4.5}$$

In summary, we have succeeded in realizing the symmetric and antisymmetric tomographic probability distributions associated with a system of  $N$  identical particles, equation (4.3), in terms of the integral kernel (4.2). This result fits into the self-consistent scheme outlined at the beginning of this section in the sense of yielding a realization of the permutation group on the space of tomograms which does not require additional structures to be defined. In the next section, we see clearly how it works on an example and we check the invertibility recovering the well-known expressions for the symmetric and antisymmetric density matrices associated with a system of oscillators.

### 5. Application

Let us consider as an example the case of two independent harmonic oscillators. For this system, the time-evolution equation (3.6) has been solved in [26] and the solutions have the following expression

$$w_{nm}(x, \mu, \nu)_2 = \frac{e^{-y_1^2 - y_2^2}}{\pi |r_1 r_2| n! m! 2^{n+m}} H_n^2(y_1) H_m^2(y_2) \tag{5.1}$$

where  $r_j = e^{i\theta}(\mu_j + i\nu_j)$ ,  $y_j = x_j/|r_j|$ ,  $j = 1, 2$  and  $H_n(y)$  and  $H_m(y)$  are the Hermite polynomials of order  $n$  and  $m$ , respectively.

For the two-dimensional case, the kernel may be obtained in closed form by integrating equation (3.20) over the  $z, z'$  and  $y, y'$  variables. After some lengthy calculation, we are able to write

$$\begin{aligned} \tilde{K}_{\pm}(\xi, \mu, \nu)_2 &= f(\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2) + f(\xi_2, \mu_2, \nu_2; \xi_1, \mu_1, \nu_1) \\ &\pm [g(\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2) + g(\xi_2, \mu_2, \nu_2; \xi_1, \mu_1, \nu_1)] \end{aligned} \tag{5.2}$$

with

- $f(\xi_2, \mu_2, \nu_2; \xi_1, \mu_1, \nu_1) = \frac{k_2}{4} \delta(m_1 - \frac{\mu_1 n_1}{\nu_1}) \delta(m_2 - \frac{\mu_2 n_2}{\nu_2}) \exp\{i(x_1 + x_2 - \frac{n_1}{\nu_1} \xi_1 - \frac{n_2}{\nu_2} \xi_2)\}$ ;
- $g(\xi_2, \mu_2, \nu_2; \xi_1, \mu_1, \nu_1) = \frac{k_2}{2\pi} \frac{\nu_1 \nu_2}{|\mu_2 \nu_1 - \mu_1 \nu_2|} \exp\{\frac{1}{2} v^T \cdot A \cdot v + B \cdot v + i(x_1 + x_2)\}$ ;
- $v = \{m_1, n_1, m_2, n_2\}$ ;

$$A = \begin{pmatrix} i \frac{\nu_1 \nu_2}{(\mu_1 \nu_2 - \mu_2 \nu_1)} & i \frac{\mu_2 \nu_1 + \mu_1 \nu_2}{2(\mu_2 \nu_1 - \mu_1 \nu_2)} & 0 & 0 \\ i \frac{\mu_2 \nu_1 + \mu_1 \nu_2}{2(\mu_2 \nu_1 - \mu_1 \nu_2)} & i \frac{\mu_1 \mu_2}{(\mu_1 \nu_2 - \mu_2 \nu_1)} & 0 & 0 \\ 0 & 0 & i \frac{\nu_1 \nu_2}{(\mu_1 \nu_2 - \mu_2 \nu_1)} & i \frac{\mu_2 \nu_1 + \mu_1 \nu_2}{2(\mu_2 \nu_1 - \mu_1 \nu_2)} \\ 0 & 0 & i \frac{\mu_2 \nu_1 + \mu_1 \nu_2}{2(\mu_2 \nu_1 - \mu_1 \nu_2)} & i \frac{\mu_1 \mu_2}{(\mu_1 \nu_2 - \mu_2 \nu_1)} \end{pmatrix};$$

$$B = \left\{ i \frac{\nu_1 \xi_2 - \nu_2 \xi_1}{\mu_1 \nu_2 - \mu_2 \nu_1}, i \frac{\mu_2 \xi_1 - \mu_1 \xi_2}{\mu_1 \nu_2 - \mu_2 \nu_1}, i \frac{\nu_1 \xi_2 - \nu_2 \xi_1}{\mu_1 \nu_2 - \mu_2 \nu_1}, i \frac{\mu_2 \xi_1 - \mu_1 \xi_2}{\mu_1 \nu_2 - \mu_2 \nu_1} \right\}$$

where  $v^T$  denotes the transposed vector and  $\cdot$  is the usual matrix product, rows by columns. This is all we need to obtain the symmetrized and antisymmetrized MDFs corresponding to all the states of the two oscillators, although equation (3.19) is in general not easy to integrate.

For the first non-trivial case, i.e. the first excited state,  $w_{\pm}^{01}(\xi, \mu, \nu)_2$ , we obtain, substituting equations (5.2) and (5.1) for  $w^{01}$  in equation (3.19)

$$\begin{aligned}
w_{\pm}^{01}(\xi, \mu, \nu)_2 &= \int D_2(x, m, n) \tilde{K}_{\pm}(x, \mu, \nu | x, m, n)_2 \frac{e^{-\nu_1^2 - \nu_2^2}}{\pi |r_1 r_2| n! m! 2^{n+m}} H_0^2(\nu_1) H_1^2(\nu_2) \\
&= \frac{\exp\left(-\frac{\xi_1^2}{\mu_1^2 + \nu_1^2} - \frac{\xi_2^2}{\mu_2^2 + \nu_2^2}\right)}{\pi \sqrt{(\mu_1^2 + \nu_1^2)(\mu_2^2 + \nu_2^2)}} \left[ \left(2 \frac{\xi_2}{\sqrt{\mu_2^2 + \nu_2^2}}\right)^2 + \left(2 \frac{\xi_1}{\sqrt{\mu_1^2 + \nu_1^2}}\right)^2 \right. \\
&\quad \left. \pm 2\xi_1 \xi_2 \left( \frac{\mu_1 \mu_2 + \nu_1 \nu_2}{(\mu_1^2 + \nu_1^2)(\mu_2^2 + \nu_2^2)} \right) \right]. \tag{5.3}
\end{aligned}$$

As can be seen, only the first two terms of the sum are probability distributions, whereas the last term is an interference term.

For our edification we can check, using the inverse formula (3.17), that we obtain the correct expression for the symmetrized and antisymmetrized density matrices associated with a system of two harmonic oscillators, that is

$$\rho_{\pm}^{01}(x, x')_2 = \frac{1}{4} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2 + x_1'^2 + x_2'^2)\right\} [4x_2 x_2' + 4x_1 x_1' \pm (4x_1 x_2' + 4x_2 x_1')]. \tag{5.4}$$

## 6. Conclusions

In summary, we have studied the permutational symmetry of tomograms of quantum states for systems of identical particles of bosonic and fermionic nature. We have shown that the tomographic probability distributions of identical particles in the probability representation of quantum mechanics are associated with the realization of the action of  $G \otimes G$ , where  $G$  is the permutation group and  $\otimes$  denotes the direct product. It is demonstrated that the tomographic probability distribution, i.e. the MDF, is a sum of terms, some of them directly recognizable as tomograms, some others that are interference terms, which are not associated with a probability distribution by themselves, but only in the given combination. The main results of our analysis are summarized by equations (4.2) and (4.3), where tomograms of identical particles are explicitly derived in terms of an integral kernel.

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